WHAT CAN SEMIOTIC THEORY CONTRIBUTE TO AN ENACTIVIST ANALYSIS OF SENSE MAKING WITH MULTIPLE ARTIFACTS?

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This project seeks to understand the emergence of mathematical meanings mediated by learners' interactions with multiple artifacts. Extending our prior work which took an enactivist approach and revealed the dynamics of embodied interactions fundamental to understanding fraction division, we now employ a semiotic lens to illuminate how learners make personal meanings from their engagement with multiple artifacts and translate them into more generalized mathematical meanings. We are doing so by taking a semiotic approach to tracking the emergent phenomenon of two learners' meaning making as it arises from the complex interplay of signs. We rely on our findings to argue that semiotic theory can be used as a resource to complement and enhance an enactive analysis of the unfolding of sense making with multiple artifacts. Implications for the design of learning experiences with multiple artifacts are proposed.

Keywords: Mathematical Representations, Cognition, Learning Theory, Rational Numbers

Hiebert and Grouws (2007) synthesized evidence from a number of studies to argue that the conceptual learning of mathematics is associated with teachers' and students' "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (p. 391). Indeed, the Principles and Standards laid out by the National Council of Teachers of Mathematics (NCTM, 2000) emphasize the value of representing mathematical ideas in a variety of ways, and that these representations are fundamental to how we understand and apply mathematics. Its more recent Principles to Actions (NCTM, 2014) identifies, "Use and connect mathematical representations" as one of only eight "high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (p. 9). Much research has been done regarding the ways in which teachers can support students' engagement with multiple representations. What is less well understood is the process by which multiple representations of a concept can be leveraged in order to contribute to learners' meanings for concepts. The better we understand this process, the better prepared we are to *engage and support all learners.*

Findings from an enactivist analysis of strategy development in mental mathematics contexts suggest that the nature of the processes at play cannot be understood in terms of solvers' flexibility and choice of strategies (Proulx, 2013). Rather, these processes are dynamic, emergent, and contingent on "an ongoing loop" (p. 319) of interactions between the problem and the solver. Since meaning making results from problem solving, and since problem solving has been shown to be dynamic, emergent, and contingent (Proulx, 2013), it would seem that meaning making might be, as well. Furthermore, research has also shown that mathematical meaning making is inextricably linked to the material and symbolic tools that mediate its learning (Verillon & Rabardel, 1995). Following these lines of thought, in our prior work (Greenstein et al., 2021) we sought to determine what an enactivist analysis could reveal about the nature of

processes at play involved in mathematical meaning making in the context of multiple artifacts. We did so through the analysis of the mathematical activity of a father and daughter as they aimed to make sense of an algorithm for fraction division using a manipulative designed for learners' engagement with fraction concepts. What that work revealed was the potential power of manipulatives to provoke particular forms of embodied experience (Abrahamson & Sánchez-García, 2016; Nathan 2021) fundamental to the conceptual learning of mathematics. It also revealed that although students' use of multiple representations is increasingly prevalent in their learning of mathematics, the process by which learners connect these representations is actually quite complex (Greenstein, 2013). That process can be supported by teachers' and students' commitments to the integrity of a model of knowing as making sense, and to the availability of artifacts such as manipulatives that provide learners with immediate feedback they can use to self-assess the viability of their reasoning.

This work seeks to complement the findings of the benefits of these embodied experiences by further illuminating how learners make personal meanings from their interaction with multiple representations of a concept and translate those meanings into more generalized mathematical meanings. We do so by employing a semiotic lens to track the evolution of these emergent meanings as we address the question, "*What can a semiotic lens contribute to an enactivist perspective on fraction division meaning making with multiple artifacts?*"

Theoretical Framework

Two perspectives ground this study, the enactivist theory of cognition (Maturana & Varela, 1987; Varela, et al, 1992) and the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). We first use enactivism to understand the evolutionary dynamics of two learners' embodied interactions as they aim to make sense[3] of fraction division using a manipulative designed for the exploration of fraction concepts. Then, we take a semiotic approach to analyzing these interactions in order to elucidate how the personal meanings that emerge from their engagement with each artifact are coordinated and ultimately converge upon a mathematical meaning of fraction division.

Two principles underlie the enactive approach: "1) Perception consists in perceptually guided action, and 2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al., 1992, pp. 172-173). Thus, from this perspective, knowing is not "outward manifestations of some inner workings" (Davis, 1995, p. 4) but a dynamically co-emergent phenomenon that arises and is made visible within a world of significance brought forth (Maturana & Varela, 1987) through one's goal-directed, "embodied (enacted) understandings" (Davis, 1995, p. 4).

By viewing knowing in the interactivity of learners (Davis, 1995), the enactivist perspective offers an alternative to the conventional view of knowledge as a static accumulation of facts, strategies, and ideas. Cognition, or active knowing, is "not something that happens 'inside' brains, bodies, or things; rather, it emerges" in the interactions between them (Malafouris, 2013, p. 66). In Davis's (1995) adaptation of Maturana and Varela's (1987) words, "*Knowing* is *doing* is *being*" (p. 7). We can consider one's way of knowing/doing/being as driven by the evolutionary imperative for an organism to maintain a functional, or harmonious, relationship

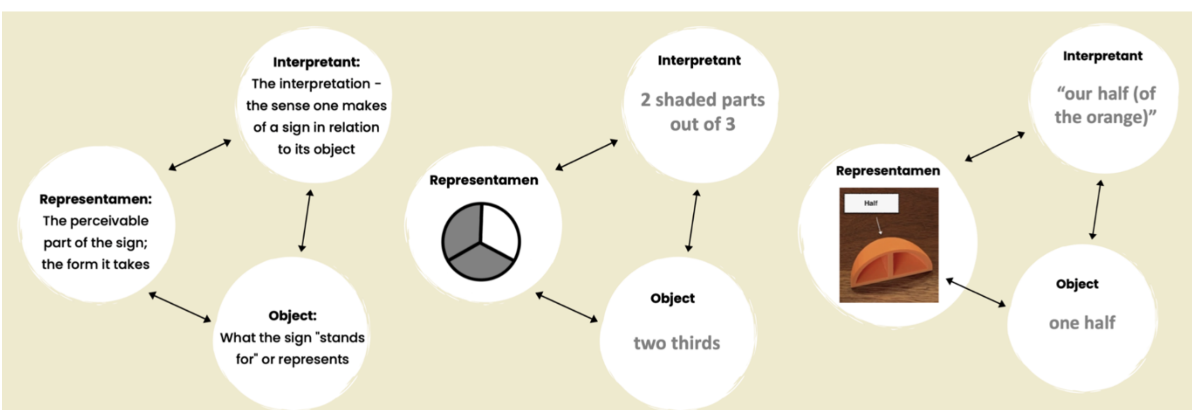
[3] We tend to use "sense making" in the context of our enactivist analysis, since cognition is defined in relation to our senses. We tend to use "meaning making" elsewhere in its conventional sense. However, we will also intentionally use them interchangeably to endorse the enactivist view that the terms are synonymous.

with its environment. “In other words, to know is to respond adequately; it is a doing that *fits* into the context where it emerges as the organism (e.g., a student, a researcher) interacts effectively with their environment” (Maheux & Proulx, 2015, p. 212). This drive toward a “good enough” solution rather than an optimal one is theoretically linked to the concepts of structural coupling and structural determinism.

Structural coupling is associated with the Darwinian concept of co-evolution, whereby an organism and its environment experience mutual structural changes through recursive and repeated interactions (Maturana & Varela, 1987) that allow their co-evolution to continue. Thus, their fit is dynamic and contingent upon their unique histories of recurrent interactions and structural changes (Maturana, 1988, as cited in Reid & Mgombelo, 2015, p. 175). So, as the environment provides a source of “triggers” that occasion the evolution of the organism, that evolution is determined by the organism’s structure, a phenomenon that Maturana and Varela (1987) conceive of as *structural determinism*.

As mentioned earlier, in our prior work we took this enactivist perspective on mathematical activity as knowing-in-action to investigate the in-the-moment problem solving of a father and daughter as they seek to understand fraction division. From this perspective, their understanding is made evident as they seek to find harmony in mathematical meanings across initially conflicting interpretations of the various elements of two artifacts: 1) the paper-based, “flip-and-multiply” algorithm for fraction division, and 2) a manipulative that the daughter designed for engagement with fraction concepts. As Maffia and Maracci (2019) have found, this use of multiple artifacts within problem solving contexts can elicit the production, transformation, and coordination of personal meanings (that resolve dissonances into harmonies) toward more culturally established mathematical signs.

In order to understand how meanings emerge from contexts in which multiple artifacts mediate one’s meaning making, we leverage Maffia and Maracci’s (2019) concept of *semiotic interference,* which relies on Presmeg’s (2006) use of Peirce’s (1998) triad of sign relations. As depicted in Figure 1 (left), Peirce’s *sign* is a triadic relationship among a *representamen* (the perceivable part of a sign; the perceiver’s personal representation[4]), an *object* (what the sign stands for), and an *interpretant*, which “is the result of trying to *make sense* of the relationship… [between] the object and the representamen” (Presmeg, 2006, p. 170, emphasis added).



[4] This conception of “representation” is distinct from its typical use in mathematics education (e.g., tables, graphs, expressions, manipulatives; NTCM, 2000, 2014). Thus, in the context of our semiotic analysis, we will use “representation” in the Peircean sense (1998) and “artifact” in the sense that it is typically used in our field.

Figure 1: The triadic relationship of a sign (adapted from Peirce, 1998, and Presmeg, 2006); the example of a sign for “two-thirds”; and an example using a piece of the Fraction Orange, presented in Figure 3 below.

Now, semiotic interference describes the phenomenon that arises whenever “the *interpretant* of a *sign* whose *object* belongs to the context of [one] artifact is translated by a student in a new sign whose object belongs to the context of another artifact” (Maffia & Maracci, 2019, p. 3-58; see Figure 2, right). Thus, semiotic interference provides a window into learners’ chaining of signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) as they negotiate their interpretations in order to converge upon a meaning for fraction division. Presmeg (2006) envisions a semiotic chain as a nested relationship of signs (see Figure 2, left) such that once a sign is established (e.g., S_1), it can then be regarded as the object of a new sign. This new sign (e.g., S_2) “comprises everything in the entire chain to that point” (p. 169). The ongoing process of enchaining signs (see Figure 2, center) is meant to depict the emergent phenomenon of meaning making that arises from this “complex interplay of signs” (Maffia & Maracci, 2019, p. 3-57).

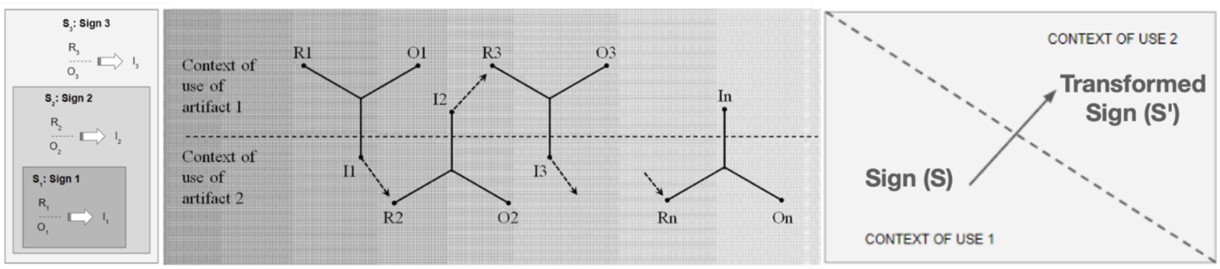


Figure 2: Left: Depiction of Presmeg’s nested model of semiotic chaining (2006, p. 171); Center: Semiotic interference in a semiotic chain (Maffia & Maracci, 2019, p. 3–59); Right: Depiction of semiotic chaining across contexts.

Methods

This project is part of a larger study (Greenstein & Seventko, 2017; Greenstein & Olmanson, 2018; Greenstein et al., 2019; Greenstein et al., 2020; Greenstein et al., 2021; Akuom & Greenstein, 2022) that tested the hypothesis that a pedagogically genuine Making experience (Halverson & Sheridan, 2014) of a physical manipulative for mathematics learning would be formative for the development of practicing and prospective mathematics teachers’ (PMTs’) inquiry-oriented pedagogy. The study took place in a specialized mathematics course for PMTs at a university in the northeastern United States. For the project reported here, we took a revelatory case study approach (Yin, 2014) in order to determine what an enactivist-semiotic perspective might reveal about the problem-solving activity of “Dolly” and her father, “Lyle” (both pseudonyms).

Dolly was a participant in the larger study. She calls the manipulative she designed a “Fraction Orange” (Figure 3) and in designing it, she aimed to create a tool embedded with mathematical meanings that afford the mediated learning of fraction concepts (e.g., part-whole meaning of fractions, measurement meaning of division). The orange is a sphere partitioned into two hemispheres. One hemisphere is further partitioned into fourths, eighths, and sixteenths of the whole, and the other is further partitioned into thirds, sixths, and twelfths. Dolly designed the fraction orange to support a student’s learning of fraction division. However, she and the

instructor (first author on this paper) agreed that an adult would be fine, too. Many adults have not yet had the privilege or experienced the joy of making sense of fraction division.

Dolly's manipulative and the thirteen-minute problem-solving interview she conducted with Lyle constitute the data for this case study. Three researchers, including Dolly, analyzed the data both individually and in collaborative dialogue. We undertook the analysis by analyzing "verbal utterances through line-by-line analysis of the transcripts; stud[ying] body language and intonation by viewing video tapes...; and inferr[ing] mathematical forms and objects from the participants' actions, utterances and notations" (Simmt, 2000, p. 154), which constitute Dolly and Lyle's knowings-in-action as they aimed to coordinate meanings for fraction division in the manipulative and in an algorithm that ostensibly substantiates those meanings.

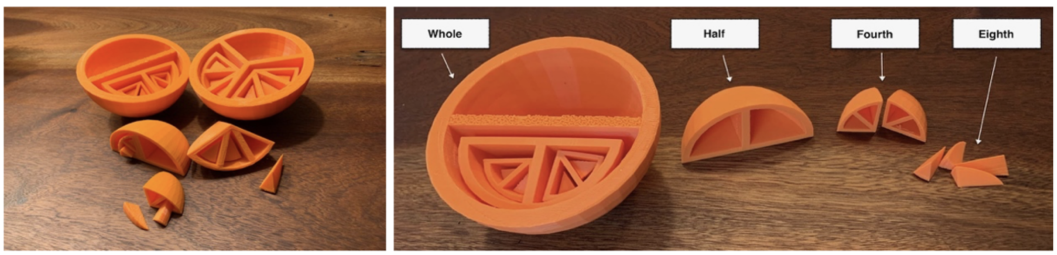


Figure 3: Dolly's Fraction Orange

Findings

Here we present excerpts from our semiotic analysis of the data, which we undertook in order to assess its viability for complementing an enactive analysis that revealed the interactions that were fundamental to two learners' conceptual learning of fraction division. The intent of this complementary analysis is to understand and depict *how* their mathematical meanings were achieved through those interactions.

The Initial Emergence of Personal Mathematical Meanings

At the outset of the interview, Dolly asks Lyle to solve the problem, ½ ÷ ¼, which she presents to him on paper alongside her fraction orange (Figure 4a). Although both the orange and the pen and paper are made available to him, Lyle chooses the pen, performs the standard, flip-and-multiply algorithm, and declares his answer to be 2, having simplified the 4/2 to 2/1. With enactivist principles framing our interpretations, we suggest that these are structurally-determined actions informed by Lyle's history of structural coupling with traditional school mathematics, where the answer derived from solving a fraction-division task by flipping and multiplying was deemed good enough to "survive." It constituted what Lyle needed to do to achieve harmony within the milieu of mathematics classroom environments.

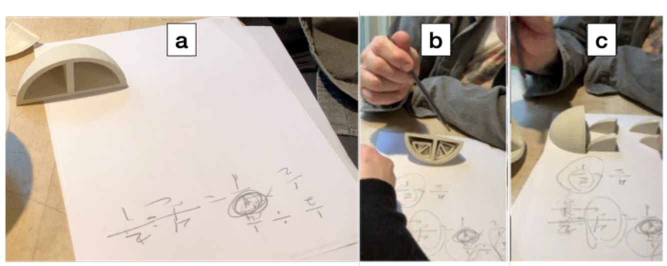


Figure 4: a) The Algorithm, b) the Half-Piece, and c) the Fourth-Pieces

Next, because Dolly is conducting the interview in order to assess the efficacy of her manipulative, she points to her fraction orange and asks Lyle, *"Can you show me with this?"* It is at this moment where we observe (a) the launching point on an emergent path of non-linear and unfolding (Proulx, 2019) problem-solving interactions bounded by alternating moments of what we refer to as *harmony* (a pleasing fit) and *dissonance* (a displeasing conflict or lack of fit), and (b) the emergence and evolution of personal meanings toward established mathematical meanings as the learners strive to harmonize their interpretations across the two artifacts: pieces of the orange and symbolic forms on paper. Mindful of the view of cognition as effective action from an enactivist perspective, we deliberately chose "harmony" and "dissonance" to reflect the cognitive and affective constitution of moments of fit or lack thereof, respectively. In the excerpt below we observe Lyle's enacted responses to the task Dolly posed to him.

Lyle: A half divided by a quarter… <*removes what he considers to be a half-piece*> a half divided by a quarter <*he counts the 4 fourth-pieces inside the 2 sections of the half-piece* (see Figures 4b and 4c)> is four.
Dolly: <*pointing to Lyle's written work*> But that's not what you got.
Lyle: Uh oh. <*Lyle pulls out the 4 fourth-pieces and looks back-and-forth between the paper and the orange as a timbre of concern grows in his voice:*>. Uh oh. A half, <*Lyle points to the now-empty half-piece.*> divided by a quarter. Why doesn't that work?

We see this as a moment of dissonance in meanings originating from Lyle's contrasting interpretations of his representations of the orange (i.e., manipulatable pieces inside pieces) and the algorithm (i.e., manipulatable symbols in expressions; Landy & Goldstone, 2007). Although he declares that the answer "is four," the somatic marker of his facial expression conveys the dissonance he was experiencing. When pressed for an explanation – *"But that's not what you got"* – a verbal marker of that dissonance is expressed: *"Uh oh…Why doesn't that work?"* Apparently, Lyle's knowing of fraction division as he actively perceives it (i.e., his engagement is both active and perceptually guided; Harvey, 2017) in the algorithm is discordant with his knowing of fraction division as he actively perceives it in the fraction orange.

The Messiness of Multiple Representations

In this next exchange, Dolly is surprised to find that she's not sure of her meanings, either. Thus, we observe both father and daughter striving for harmony in meanings as they attempt to make sense of the fraction division expression, ½ ÷ ¼.

Dolly: Here's our half. <*She takes the half-piece, places it next to the algorithm carried out on paper, and asks Lyle:*> And how many quarters go into a half?
Lyle: <*Looking at the orange*> Two. <*Shifting his attention to the written work and restating that outcome:*> Four. <*Shifting his attention back to the orange, and then back again to the paper:*> Is that half of a quarter, though? It's half <*pointing to the ½ in the expression, ½ ÷ ¼*> of a quarter. <*now pointing to the ¼ on the paper*> It's not half of a whole thing.
Dolly: <*Sensing that Lyle is still deliberating:*> It's a quarter of a half, right?
Lyle: <*Lyle refers to the orange, then to the paper, and then back to the orange. With uncertainty:*> Yeah?
Dolly: How many quarters of a half are there? <*laughs and smiles*> Why is this so hard?

Here, Dolly begins by enacting her interpretation of the object "one half" as a half-piece of the orange (her representamen) and physically placing the piece on the paper, as if to propose a

common meaning between them. Thus, her interpretation of the posed problem, ½ ÷ ¼, as *"How many quarters go into a half?"* is both meaningful and en-actionable to her. Lyle, referencing the orange and evoking his own interpretations of the objects "one fourth" and "one half," determines that two quarter-pieces fit into one half-piece (his representamen) and declares the answer to be "2." Immediately thereafter, however, he shifts his attention to the algorithm on the page and changes his answer to "4," presumably trying to match the "4" in the "4/2" that is the outcome of the algorithm and that at one point he calls "4 halves." In doing so, he seems to privilege the algorithm over the embodied actions that are reflective of interpretations he attributed to representamens in the orange. Next, he checks his answer of "4" against his interpretation of the division expression: *"Is that half of a quarter, though?"* Then, Dolly steps in to suggest a different interpretation of the expression: *"It's a quarter of a half, right?"* Lyle's unsure: *"Yeah?"* To which Dolly responds with yet another interpretation presumably informed by Lyle's: *"How many quarters of a half are there?"*

Lyle's actions at this moment are directed at finding harmony in meanings across multiple interpretations of multiple elements in the fraction division expression and the two different values he derives from it. The dissonance Dolly's feeling is expressed through her utterance, *"Why is this so hard?"* Her dad is feeling it, too. Their collective words and actions express the messiness of their engagement with incompatible interpretations of multiple representations and what it feels like as they strive to make a generalizable mathematical meaning for fraction division through what amounts to a complex interplay of signs.

By this point we've seen the value of using a semiotic lens to track the complex interplay of Dolly and Lyle's multiple interpretations and the evolution of emergent meanings. We share this next excerpt in order to demonstrate the utility of a semiotic depiction of the enchaining of signs. Here, Dolly and Lyle find harmony in their meanings for fraction division by resolving some of the many dissonances that were swirling around this last excerpt.

Traversing the Semiotic Landscape: The Enchaining of Signs

This excerpt took place in the final moments of Dolly and Lyle's problem solving. Just prior to this moment, they enchained their signs for – and thus made sense of – ½ and ¼ by translating their interpretations of representamens in the orange to elements of the algorithm. Now, they are engaged in similar meaning-making activity as they aim to find interpretations for the ½ and ¼ in the posed problem, ½ ÷ ¼.

> *Dolly:* We wanna take a half of one and divide it by a quarter of one, right?
> *Lyle:* Yes.
> *Dolly:* Take a half of one and divide – oh, that's what it is! … We wanna take this <*half-piece of the orange*> and see how many of those <*quarter-pieces*> fit in there <*the half-piece. Then, confidently:*>And that's why our answer is 2.
> *Lyle:* Yes.
> *Dolly:* There's still two halves in a whole, 'cuz this <*the expression, ½ ÷ ¼*> is in regards to a whole. <*rephrasing*> This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half.
> *Lyle:* Yeah. It makes sense that way.

In this excerpt, we observe the meaning Dolly makes of the expression, ½ ÷ ¼, by enchaining interpretations of ½ and ¼ according to the measurement meaning of division she and Lyle enacted earlier. Next, Lyle proceeds to re-enact the interpretation for himself.

Lyle: <pointing with his pencil to ½ on the page:> So this is half of a whole *<then pointing to the ¼ on the page.>* and this is a quarter of a whole. *<Next, he turns his attention to the orange (as in Figure 4b) and points to the half-piece:>* Half of a whole. *<Next, he points to each quarter-piece in a sweeping motion across the pieces:>* Quarter of a whole *<Then, pointing to the two quarter-pieces, he continues:>* is 2. *<Thus, he appears to be establishing that the number of quarter-pieces he's identified, 2, is the answer to the posed problem, ½ ÷ ¼>.*

Dolly: <pointing to the 2 quarter-pieces and agreeing with Lyle> Yeah, 'cause there's two quarters of a whole.

Lyle: *<with a sigh of relief>* Yeah, that makes sense.

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly has just done, Lyle uses the pencil to re-enact a physical bridge between the elements of the problem and the pieces of the orange. We interpret his subsequent actions as an effort to match his interpretations of half of a whole and quarter of a whole in the symbolic expression to representamens he's identified in the orange (the half-piece and the quarter-piece, respectively; see Figure 5, left). These actions reify the harmony that has finally emerged from recursive interactions that culminate in an enchaining of signs signifying the sense he and Dolly have made to form a newly coupled structure of fraction division (see Figure 5, right).

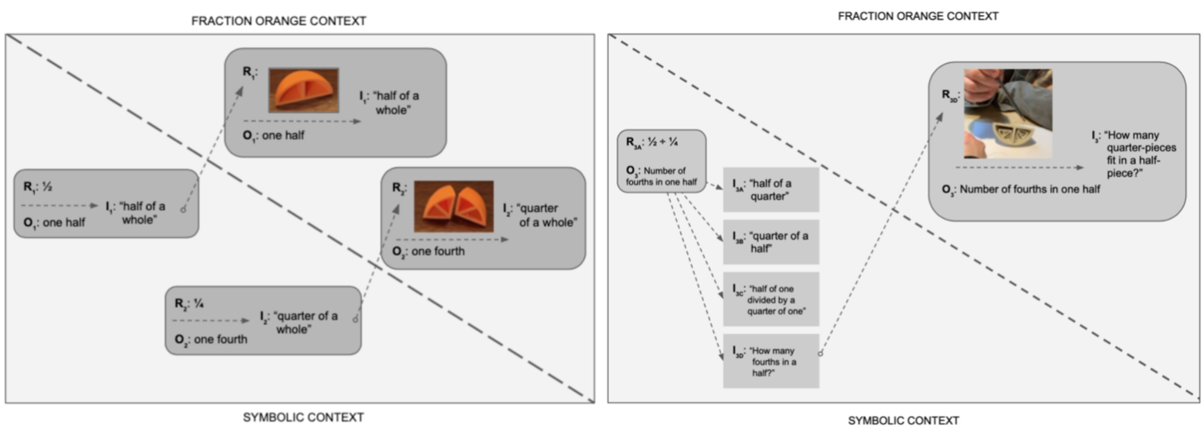


Figure 5: Enchained and unchained signs for one half and one fourth (Left), and for the number of fourths in one half (Right)

Concluding Discussion

A taken-as-shared principle of mathematics education is that the learning of mathematics entails using and connecting multiple representations of mathematical ideas (e.g., Hiebert & Grouws, 2007). This work presented here reminds us that learners' paths of problem solving are contingent on their particular mathematical structures and interactions (Proulx, 2013), and thus the process by which learners connect these representations is actually quite complex (Greenstein, 2013; Lesh et al., 1987; Maracci & Mariotti, 2019). In order to make sense of this complexity and thereby offer implications for the designs of tools and tasks that have the potential to support the evolution of all learners' mathematical meanings, this work set out to discern the contributions of a semiotic lens to an enactivist analysis of fraction division meaning making with multiple artifacts. That enactivist analysis revealed the mediated interactions with multiple artifacts that were productive for developing mathematical meanings for fraction

division. The findings presented here in the depictions of Dolly and Lyle's chaining of signs through semiotic interference revealed just how their mathematical meanings were achieved through the interactions revealed by the enactivist analysis. In other words – and this is the contribution we propose to have made – the semiotic analysis was successful for explicating the interpretive constitution of two learners' enacted meanings. Thus, we take these findings to argue that semiotic theory can be used as a resource to complement and enhance an enactive analysis of the unfolding of meaning making with multiple artifacts.

Acknowledgments

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